Section 4.5: Row and column spaces

New vocabulary:

- row space, column space of a matrix.
- row rank, column rank, rank
- pivot column
- null space

We learn:

- an algorithm to find a basis for the column space
- an algorithm to find a basis for the row space
- how to find a subset of a spanning set that is a basis
- how to extend an independent set to a basis

Question like Section 4.5 13-16 (and also 1-12)

Find a subset of the vectors

$$\begin{bmatrix}1\\2\\-1\end{bmatrix} \quad \begin{bmatrix}2\\4\\-2\end{bmatrix} \quad \begin{bmatrix}3\\1\\0\end{bmatrix} \quad \begin{bmatrix}4\\3\\-1\end{bmatrix}$$

that is a basis for the subspace they span.

Overview of the algorithm.

Take the first vector. If it is non-zero, keep it.

Is the second vector dependent on the vectors we have so far (i.e. a scalar multiple of the first).

If dependent, throw it out

If not, keep it.

Is the 3rd vector dependent on the first two?

Yes — throw it

No — keep it,

We end with a set of vectors that is independent with the same span as the original vectors.

Question like Section 4.5 13-16 (and also 1-12)

Find a subset of the vectors

$$\begin{bmatrix}1\\2\\-1\end{bmatrix} \quad \begin{bmatrix}2\\4\\-2\end{bmatrix} \quad \begin{bmatrix}3\\1\\0\end{bmatrix} \quad \begin{bmatrix}4\\3\\-1\end{bmatrix}$$

that is a basis for the subspace they span.

Solution: Is vector 2 dependent on vector 1?

Solve $x_1 \begin{bmatrix}1\\2\\-1\end{bmatrix} = \begin{bmatrix}2\\4\\-2\end{bmatrix}$

Reduce $\begin{bmatrix}1 & 2 & 3 & 4\\2 & 4 & 1 & 3\\-1 & -2 & 0 & -1\end{bmatrix}$

$② \to ② - 2①$
$③ \to ③ + ①$
$\begin{bmatrix}1 & 2 & 3 & 4\\0 & 0 & -5 & -5\\0 & 0 & 3 & 3\end{bmatrix}$

We can solve the equation for x_1. It is consistent. The second vector is dependent on the first. We throw it.

Is vector 3 dependent on 1st 2?

No: The row 0 0 -5 ⋮ shows the eqns are inconsistent. Keep vector 3.

More reduction. $\begin{bmatrix}1 & 2 & 3 & 4\\0 & 0 & -5 & -5\\0 & 0 & 0 & 0\end{bmatrix}$

Is vector 4 dependent on first 3?

Yes. Throw it. $\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}$ are a basis for this space.

Definition: the column space of a matrix A is

is the span of the columns of the matrix.

Example: the column space of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ -1 & -2 & 0 & -1 \end{bmatrix}$$

is the space spanned by the vectors

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$$

Definition: a pivot column of a matrix A is a column of A for which the echelon form of A has a leading entry (or pivot).

Example: The pivot columns of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ -1 & -2 & 0 & -1 \end{bmatrix}$$

are cols 1 & 3.

Theorem.
The pivot columns of a matrix A form a basis for the column space of A.

Done

Example: The vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3$ are a basis for the column space of A.

Questions:

1. What is the dimension of the column space of the matrix A in the example?
   a. 2 ✓ The basis has 2 elements.
   b. 3
   c. 4

2. Of what space is the column space of A a subspace?
   a. $\mathbb{R}^2$
   b. $\mathbb{R}^3$ ✓
   c. $\mathbb{R}^4$

Pre-class Warm-up!!! 🌷

What is the dimension of the column space of the following matrix?

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

a. 0

b. 1

c. 2 ✓

d. 3

e. 4

When did we learn the meaning of the word 'basis' for a vector space V?

a. This week

b. Last week

Definition. The row space of a matrix A is the span of the rows of A.

Example. The row space of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ -1 & -2 & 0 & -1 \end{bmatrix}$

is the subspace of R^4 spanned by

$[1\ 2\ 3\ 4]$, $[2\ 4\ 1\ 3]$ and $[-1\ -2\ 0\ -1]$

In the book the row space is spanned by

$(1\,2\,3\,4) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $(2\,4\,1\,3) = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}$ and $(-1\,-2\,0\,-1) = \ldots$

Look at the row space of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
(It's quicker to write down!)

It is the set of all vectors

$a\,[1\ 2] + b\,[3\ 4]$ where $a, b \in \mathbb{R}$

Put it in echelon form $② \to ② - 3①$ $\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$

This has row space all vectors

Theorem 2. The row space of A is not changed by elementary row operations. Hence A has the same row space as its echelon form.

$c\,[1\ 2] + d\,[0\ -2]$

These two sets of vectors are the same

because $[0\ -2] = [3\ 4] - 3\,[1\ 2]$

so $c[1, 2] + d[0\ -2] = c[1\ 2] + d([3\ 4] - 3[1\ 2])$

$= (c - 3d)[1, 2] + d[3\ 4]$

Similarly we can write $a\,[1\ 2] + b\,[3\ 4]$

as a linear combn of $[1\ 2]$, $[0\ -2]$ because

$[3, 4] = [0, -2] + 3\,[1\ 2]$

$a\,[1\ 2] + b\,[3\ 4] = a\,[1\ 2] + b([0, -2] + 3[1, 2])$

Question: true or false in general?:

'Let A be a matrix. The column space of A is not changed by elementary row operations. Hence A has the same column space as its echelon form.'

True False ✓

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Theorem 2. The row space of A is not changed by elementary row operations. Hence A has the same row space as its echelon form.

Question like Section 4.5, 1-12

Find a basis for the row space of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ -1 & -2 & 0 & -1 \end{bmatrix}$$

Solution. Put it in echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The non-zero rows are now a basis for the row space.

They span (row space is unchanged)

They are independent: if

$$a\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & -5 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a, & 2a, & 3a-5b, & 4a-5b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

$$a = 0, \quad 3a - 5b = 0, \quad b = 0.$$

This row space has dimension 2 as a subspace of $\mathbb{R}^4$.

Theorem 2 extra: The non-zero rows of the echelon form of A form a basis for the row space of A

Definition. The row rank of a matrix A is the dimension of its row space.

The column rank of a matrix A is the dimension of its column space.

Theorem. For any matrix A the row rank and column rank are equal.

Example. For the matrix
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ -1 & -2 & 0 & -1 \end{bmatrix}$$
They are both 2.

Definition. The common value of the row rank and column rank of a matrix is called the rank of the matrix.

Proof of theorem:

row rank = number of non-zero rows in the echelon form.

column rank = number of leading entries in the echelon form.

These are the same.

Question like Section 4.5, 17-20.

Find a basis for R^3 that contains the vectors

$$\begin{bmatrix}3\\2\\-1\end{bmatrix} \text{ and } \begin{bmatrix}2\\-2\\1\end{bmatrix}$$

Solution. (Check these vectors are independent)

Adjoin $\begin{bmatrix}3\\2\\-1\end{bmatrix}, \begin{bmatrix}2\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

to get a spanning set for $\mathbb{R}^3$.

Find an independent subset, Reduce

$$\begin{bmatrix}3 & 2 & 1 & 0 & 0\\2 & -2 & 0 & 1 & 0\\-1 & 1 & 0 & 0 & 1\end{bmatrix} \xleftrightarrow{①-③} \begin{bmatrix}-1 & 1 & 0 & 0 & 1\\2 & -2 & 0 & 1 & 0\\3 & 2 & 1 & 0 & 0\end{bmatrix}$$

② → ② + 2①
③ → ③ + 3①

$$\begin{bmatrix}-1 & 1 & 0 & 0 & 1\\0 & 0 & 0 & 1 & 2\\0 & 5 & 1 & 0 & 3\end{bmatrix}$$

② ↔ ③

$$\begin{bmatrix}-1 & 1 & 0 & 0 & 1\\0 & 5 & 1 & 0 & 3\\0 & 0 & 0 & 1 & 2\end{bmatrix}$$

Leading entries in cols 1, 2, 4.

$\begin{bmatrix}3\\2\\-1\end{bmatrix}, \begin{bmatrix}2\\-2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ is a basis for $\mathbb{R}^3$.

Definition. The nullspace of A is the vector space of solutions to Ax = 0.
It might be written Null A .
~~It is~~ called the nullity of A.
Its dimension is called

Theorem. For any m x n matrix A,
rank A + dim Null A = n

rank A = no. leading entries . nullity = no. of free variables.

These add to the total number of columns

Example. Find a basis for the nullspace of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ -1 & -2 & 0 & -1 \end{bmatrix}$$

Question: Have we done this calculation before?

Yes No

Echelon form
$$\begin{array}{cccc} w & x & y & z \end{array}$$
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Free variables x and z

$y = -z \qquad w = -2x - 3y - 4z$
$= -2x - z$

General solution
$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2x-z \\ x \\ -z \\ z \end{bmatrix}$$

$$= x\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ is a basis.

Definition. The nullspace of A is the vector space of solutions to $Ax = 0$.
It might be written Null A .
It is called the nullity of A.

Theorem. For any $m \times n$ matrix A,
$\operatorname{rank} A + \dim \operatorname{Null} A = n$

Example. Find a basis for the nullspace of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ -1 & -2 & 0 & -1 \end{bmatrix}$$

Questions:

Are the following true or false in general for an n x n matrix A?

a. If A has rank n then Ax = b has a solution for every vector b in R^n.

True ✓ False Not enough information to say.

b. If Ax = 0 has a unique solution then Ax = b always has a solution, for every b in R^n.

True ✓ False Not enough information to say.

unique solution $\Rightarrow$ Null $A = \{0\}$

so rank $A = n$, $=$ dimension of column space.

col. space $= \mathbb{R}^n$.

Every b lies in the column. $Ax=b$ has a solution.

Question:

Suppose that A is an $m \times n$ matrix (m rows, n columns) and that $Ax = 0$ has a unique solution. Which of the following statements is sometimes false?

a. The columns of A are linearly independent.

b. The columns of A span a space of dimension n.

c. The columns of A are a basis for the space they span.

d. $m \leq n$